22M:025 Worksheet #2   **Name:**_____________________

Last time we were interested in proving the following statement:

Theorem. If $f(x)$ is an n^{th} degree polynomial, then $f(x)$ has $\leq n$ roots.

This is clear for $f(x)$ if $n = 1$ as a polynomial with degree 1 has the form $f(x) = a_1 x + a_0$ with $a_0, a_1 \in \mathcal{R}$. (See problem #1 below)

We will now prove that if every n^{th} degree polynomial has $\leq n$ roots, then every $(n+1)^{th}$ degree polynomial has $\leq n+1$ roots. Then our theorem will follow by induction (see pg. 81 in our text). First, we will require the following:

Lemma. If $f(x)$ is an n^{th} degree polynomial, then $f'(x)$ is an $(n-1)^{th}$ degree polynomial.

Proof. We may write

$$f(x) = a_n x^n + a_{n-1}x^{n-1} ... + a_1 x^1 + a_0 = \sum_{i=0}^{n} a_i x^i$$

Then see problem #2 below.

Now let $f(x)$ be an $(n+1)^{th}$ degree polynomial and assume every n^{th} degree polynomial has $\leq n$ roots. By the lemma, $f'(x)$ is n^{th} degree, and by assumption has at most n roots. Let r_i, $1 \leq i \leq m$ be the roots of $f(x)$, in other words, $r_1, r_2, ..., r_m$ are all the numbers such that $f(r_i) = 0$. What we want to show is that in fact $m \leq n+1$. Now by Rolle's Theorem, for each $i \leq m-1$ there is a point $z_i \in (r_i, r_{i+1})$ such that $f'(z_i) = 0$. Thus for all i with $1 \leq i \leq m-1$, z_i is a root of $f'(x)$. (*) Now $m \leq n+1$ (see problem #3). Thus $f(x)$ has m roots with $m \leq n+1$ so by induction, every n^{th} degree polynomial has at most n roots, as was to be shown.

Problems:

1. **(1 Point)** Show that if $f(x)$ is a polynomial with degree 1, $f(x)$ has exactly one root, and thus has ≤ 1 roots.
2. **(1 Point)** Show that if $f(x)$ is an n^{th} degree polynomial, then $f'(x)$ is an $(n-1)^{th}$ degree polynomial.
3. **(2 Points)** Fill in the gap in the proof above, in other words, use the arguments from before (*) to show that $m \leq n+1$.
4. **(1 Point)** Show that inequality (as opposed to equality) is needed in the statement of our theorem.