Worksheet #1 (5 Points)

Last time we were interested in #42 from Section 2.9 and we wanted to show that for $f(x) = x^{\frac{2}{3}}$, $f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$ for $x \neq 0$. When we get to the chain rule in Chapter 3 this will all be taken care of for us, but in the mean time we have to use what we know about limits.

We are interested in

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^{\frac{2}{3}} - x^{\frac{2}{3}}}{h}$$

I have not been able to find a simple way to do this without using some other result, so I have fleshed out the approach that the solution manual uses. We will need the following idea: (Vocabulary note: in mathematics, a **lemma** is just a small theorem, usually one used to prove some bigger theorem.)

Lemma. If $a, b \in \mathcal{R},\ a, b > 0$ and $n \in \mathcal{N}$ then

$$a - b = (a^{\frac{1}{n}} - b^{\frac{1}{n}})(\sum_{i=0}^{n-1} a^{\frac{i}{n}} b^{\frac{(n-1)-i}{n}})$$

.

Proof.

$$(a^{\frac{1}{n}} - b^{\frac{1}{n}})(\sum_{i=0}^{n-1} a^{\frac{i}{n}} b^{\frac{(n-1)-i}{n}}) = a^{\frac{1}{n}} \sum_{i=0}^{n-1} a^{\frac{i}{n}} b^{\frac{(n-1)-i}{n}} - b^{\frac{1}{n}} \sum_{i=0}^{n-1} a^{\frac{i}{n}} b^{\frac{(n-1)-i}{n}} = ...$$

$$... = \sum_{i=0}^{n-1} a^{\frac{i+1}{n}} b^{\frac{(n-1)-i}{n}} - \sum_{i=0}^{n-1} a^{\frac{i}{n}} b^{\frac{(n-1)-i+1}{n}} = a^{\frac{n}{n}} - b^{\frac{n}{n}} + \sum_{i=0}^{n-2} a^{\frac{i+1}{n}} b^{\frac{(n-1)-i}{n}} - \sum_{i=1}^{n-1} a^{\frac{i}{n}} b^{\frac{(n-1)-i+1}{n}} = ...$$

$$... = a^1 - b^1 + \sum_{i=1}^{n-1} a^{\frac{(i-1)+1}{n}} b^{\frac{(n-1)-(i-1)}{n}} - \sum_{i=1}^{n-1} a^{\frac{i}{n}} b^{\frac{(n-1)-i+1}{n}} = ...$$

$$... = a - b + [\sum_{i=1}^{n-1} a^{\frac{i}{n}} b^{\frac{(n-1)-i+1)}{n}} - \sum_{i=1}^{n-1} a^{\frac{i}{n}} b^{\frac{(n-1)-i+1}{n}}] = a - b$$

Now this looks really bad, but if you follow each step, we are really just using algebra. Note that in the third line we are just changing how we add up the first sum, and in the last line the "big nasty terms" drop since we have one big term minus the same big term. You can actually use this idea in more cases, as the following corollary suggests: (Vocabulary note: in mathematics, a **corollary** is a theorem that follows from another theorem, usually with a minimal amount of work.)

Corollary. If $a, b \in \mathcal{R},\ a, b > 0$ and $m, n \in \mathcal{N}$ then

$$a^m - b^m = (a^{\frac{m}{n}} - b^{\frac{m}{n}})(\sum_{i=0}^{n-1} a^{\frac{mi}{n}} b^{\frac{m((n-1)-i)}{n}})$$

.

Proof. Let $c = a^m$ and $d = b^m$ so that

$$a^m - b^m = c - d = (c^{\frac{1}{n}} - d^{\frac{1}{n}})(\sum_{i=0}^{n-1} c^{\frac{i}{n}} d^{\frac{(n-1)-i}{n}}) = ...$$

$$... = ((a^m)^{\frac{1}{n}} - (b^m)^{\frac{1}{n}})(\sum_{i=0}^{n-1} (a^m)^{\frac{i}{n}} (b^m)^{\frac{(n-1)-i}{n}}) = (a^{\frac{m}{n}} - b^{\frac{m}{n}})(\sum_{i=0}^{n-1} a^{\frac{mi}{n}} b^{\frac{m((n-1)-i)}{n}})$$

Now we will use the above results to find $f'(x) = \lim_{h\to 0} \frac{(x+h)^{\frac{2}{3}} - x^{\frac{2}{3}}}{h}$ for $x \neq 0$. Let $a = x + h$ and $b = x$ so that we have:

$$\lim_{h\to 0} \frac{(x+h)^{\frac{2}{3}} - x^{\frac{2}{3}}}{h} = \lim_{h\to 0} \frac{a^{\frac{2}{3}} - b^{\frac{2}{3}}}{a - b} = \lim_{h\to 0} \frac{(a^{\frac{1}{3}})^2 - (b^{\frac{1}{3}})^2}{a - b} = ...$$

$$... = \lim_{h\to 0} \frac{(a^{\frac{1}{3}} - b^{\frac{1}{3}})(a^{\frac{1}{3}} + b^{\frac{1}{3}})}{a - b} = \lim_{h\to 0} \frac{(a^{\frac{1}{3}} - b^{\frac{1}{3}})(a^{\frac{1}{3}} + b^{\frac{1}{3}})}{(a^{\frac{1}{3}} - b^{\frac{1}{3}})(\sum_{i=0}^{3-1} a^{\frac{i}{3}} b^{\frac{(3-1)-i}{3}})} = ...$$

$$... = \lim_{h\to 0} \frac{(a^{\frac{1}{3}} + b^{\frac{1}{3}})}{(\sum_{i=0}^{2} a^{\frac{i}{3}} b^{\frac{2-i}{3}})} = \lim_{h\to 0} \frac{(a^{\frac{1}{3}} + b^{\frac{1}{3}})}{(a^{\frac{0}{3}} b^{\frac{2}{3}} + a^{\frac{1}{3}} b^{\frac{1}{3}} + a^{\frac{2}{3}} b^{\frac{0}{3}})} = ...$$

$$... = \lim_{h\to 0} \frac{(a^{\frac{1}{3}} + b^{\frac{1}{3}})}{(b^{\frac{2}{3}} + a^{\frac{1}{3}} b^{\frac{1}{3}} + a^{\frac{2}{3}})} = \lim_{h\to 0} \frac{((x+h)^{\frac{1}{3}} + x^{\frac{1}{3}})}{(x^{\frac{2}{3}} + (x+h)^{\frac{1}{3}} x^{\frac{1}{3}} + (x+h)^{\frac{2}{3}})} = ...$$

$$... = \frac{((x+0)^{\frac{1}{3}} + x^{\frac{1}{3}})}{(x^{\frac{2}{3}} + (x+0)^{\frac{1}{3}} x^{\frac{1}{3}} + (x+0)^{\frac{2}{3}})} = \frac{2x^{\frac{1}{3}}}{3x^{\frac{2}{3}}} = \frac{2}{3} x^{-\frac{1}{3}}$$

So in fact $f'(x) = \frac{2}{3} x^{-\frac{1}{3}}$ for $x \neq 0$. See if you can follow each step, and show the following using these techniques:

1. If $f(x) = x^{\frac{2}{5}}$ then $f'(x) = \frac{2}{5} x^{-\frac{3}{5}}$ for $x \neq 0$. **(2 Points)**
2. **Prove.** If $a, b \in \mathcal{R},\ a, b > 0$ and $k \in \mathcal{N}$ then

$$a^k - b^k = (a - b)(\sum_{i=0}^{k-1} a^i b^{(k-1)-i})$$

(**Hint:** there is a really easy way to do this.) **(2 Points)**

3. Use #2 to find a non-decimal expression for $\frac{1000 - \frac{1}{1000}}{10 - \frac{1}{10}}$ **(1 Point)**