MATH REVIEW #1 SUPPLEMENT (MATH CAMP 2007)

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What follows is a supplement of the document labeled "MATH REVIEW #1 (short version)." Sections are provided to correspond roughly with what I intend to cover each day of Math Camp 2007. These questions are primarily designed to get you thinking, not review material.

1. Day 1

1.1. **Some Examples of Sets.**

Some familar sets should be $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{R}^n, \mathbb{C}, \mathcal{C}^0, \mathcal{C}^n, \mathcal{C}^\infty$ as well as the set of all polynomial functions with real coefficients (say $\mathbb{P}$) and linear spaces on $\mathbb{R}^n$

$$\begin{aligned}
\mathcal{V} &\equiv \{x \text{ is a letter}: \quad x \text{ is a vowel}\} \\
\mathcal{G} &\equiv \{x \text{ is a letter}: \quad x \text{ appears in the word "Cheese"}\} \\
\mathcal{A} &\equiv \{x \in \mathbb{R}: \quad x \geq 0 \text{ and } x \leq 1\}
\end{aligned}$$

(1) Some questions:
   (a) How many elements does $\mathcal{G}$ have? What is $\mathcal{V} \cap \mathcal{G}$?
   (b) Relationships between $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$?
   (c) What about $\mathbb{Z}^2$ and $\mathbb{R}^2$?
   (d) Relationships between $\mathcal{C}^0, \mathcal{C}^n, \mathcal{C}^\infty, \mathbb{P}$?
   (e) Another way of writing $\mathcal{A}$?
   (f) Is $\mathcal{A}$ convex? What about $\mathcal{B} \equiv \{x \in \mathbb{R}: \quad x > 1 \text{ and } x < 2\}$?
   (g) What is $\mathcal{A} \cap \mathcal{B}$? How about $\mathcal{D} \equiv \{x \in \mathbb{R}: \quad x > 2 \text{ and } x < 1\}$? And $\mathcal{D}^c$?
   (h) What is $\mathcal{E} \equiv (((\mathcal{A} \cap \mathcal{B})^c \cap \mathcal{B}) \cup \mathcal{A}) \cap \mathcal{Q}$? How about $\mathcal{E} \cap \mathbb{Z}$?
   (i) Is the union of two convex sets convex? (Prove or provide a counterexample.)
   (j) Is the intersection of two convex sets convex? (Prove or provide a counterexample.)

1.2. **Functions and Continuity.**

(1) Some questions:

Date: 8/21/07.

(a) Is $f(x) \equiv \pm\sqrt{x}$ a function? If g is a function is $g^{-1}(x)$ always a function? (If not provide a counterexample.)

(b) If g is a continuous function and $g^{-1}(x)$ is a function, is g^{-1} continuous? (If not provide a counterexample.)

(c) On $[0, 1]$ are monotone functions invertible? Is *strict* monotonicity involved? What about continuity?

(d) Suppose f is continuously differentiable, $f(1) = 10$ and $a_n \to 1$. Let $b_n \equiv f(a_n)$. What is $\lim_{n\to\infty} b_n$?

(e) Let f be as in (d) and let $g(x) \equiv \frac{1}{x-10}$. Suppose $f'(1) > 0$. Can you compute $\lim_{n\to\infty} g \circ f(a_n)$ (allowing for $\pm\infty$)? What if $a_n \geq 1$ for all n?

1.3. **Derivatives.**

(1) Some questions:

(a) If g is a $\mathcal{C}^1$ function on $[0, 1]$ with $g' > 0$, is g^{-1} a function? (If not provide a counterexample.) How about if just $g' \neq 0$? If $g' \equiv 0$? Can you connect this to any "famous" results?

(b) If f is differentiable at x, what are

$$(\ln f(x))' \quad (\exp\{f(x)\})' \quad (5^{f(x)})' \quad (f(x)^{f(x)})'$$

(c) Assume f, g, h are differentiable at x. What is $(f \circ g \circ h(x))'$?

(d) Assume f and f^{-1} are diffentiable functions and use $f(f^{-1}(x))' = (x)' = 1$ to derive an expression for $f^{-1}(x)'$.

(e) Suppose f is continuously differentiable on $(0, 1)$ and f is strictly increasing. Use the definition of the derivative to conclude that $f'(x) \geq 0$ for $x \in (0, 1)$.

(f) A twice continuously differentiable function f is said to satisfy the inada conditions if

$$\lim_{x\to 0} f'(x) = \infty \qquad \lim_{x\to\infty} f'(x) = 0$$

What do the inada conditions have to do with the equation $f'(x) = 25$? When considering $f'(x) = 25$, would it be useful to know something like $f'' < 0$?

(g) Construct a function f which satisfies the inada conditions. Let $g(x) \equiv f'(x)$ and find $g^{-1}(x)$. How does g^{-1} relate to (f)?