Welcome to Madison!

Some useful links to materials can be found at http://www.aae.wisc.edu/morrow/. We will be working through Rebecca Moore's problem set (answers are online) from Math Camp 2005, and some additional problems here.

Suggested Problem Schedule (Less important questions are marked with a ***):

(1) Day 1
  (a) Day 1 Problem (Below)
  (b) Problems 1,2 (for 2 you can draw Venn Diagrams if you like), 4,5,7.
  (c) Fill in the following logic table:

A	B	$(\neg A) \implies B$	$(\neg A) \bigcap B$	$A \iff B$
T	T			
T	F			
F	T			
F	F			

  (d) For what values of α and x are the following functions Strictly/Weakly: Concave? Convex? Increasing? Decreasing? (These functional forms will appear over and over again.)
    (i) $f(x) = x^{\alpha}$
    (ii) $f(x) = \alpha \ln(x)$
    (iii) $f(x) = e^{\alpha x}$
    (iv) $f(x) = \alpha x$
  (e) In general, for $\alpha > 0$, the following formula holds:

$$\sum_{i=0}^{N} \alpha^i = \frac{1-\alpha^{N+1}}{1-\alpha}$$

  Sometimes the following formula is also true:

$$\sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha}$$

  For which values of α should the second equation hold? Relate the second equation to the first.
  (f) ***The *power set* of a set A, denoted $P(A)$ is defined as the collection of all subsets of A, or rather

$$P(A) \equiv \{B : B \subset A\}$$

  Find $P(A)$ for $A = \{1, 2\}$ and $A = \{\{1\}, 2, \{3\}\}$. If A contains N elements, how many elements does $P(A)$ have?
  (g) Which of the following sets are convex?
    (i) $[0, 1]$
    (ii) $(10, 100)$
    (iii) $(0, 1) \bigcup (20, 100)$
    (iv) $(1, 2) \bigcup (2, 3)$
    (v) ***$A \bigcap B$ where A and B are convex

What follows is a puzzle to "get your wheels turning." The puzzle is taken verbatim from [1]. Martin Gardner states it as follows:

> An *unlimited* supply of gasoline is available at one edge of a desert 800 miles wide, but there is no source on the desert itself. A truck can carry enough gasoline to go 500 miles (this will be called on "load"), and it can build up its own refueling stations at any spot along the way. These caches may be of any size, and it is assumed that there is no evaporation loss.
>
> What is the minimum amount (in loads) of gasoline the truck will requite in order to cross the desert? Is there a limit to the width of a desert the truck can cross?

We will take steps to solving this problem in the sections below. Working with your fellow students in encouraged. If you think you can solve the whole problem, by all means do so. (Incidentally in that case, can you solve the problem if the truck needs to return to its starting point?) It turns out in fact this is a deceptively hard problem, so we will only take steps towards solving it. If you are very interested, there are some publications regarding it.

Day 1

In order to solve this problem, one good step would to be to specify what a solution should look like. In our case, it will be to specify the details of each trip, indexed by t, that the truck takes. On each trip, either the truck succeeds in crossing the desert or goes some distance d_t, deposits some gas g_t for refueling, and returns home to get more gas. In other words, a solution consists of filling in the following table:

Table 1. Solution Table

Trip Number (t)	Distance Gone (d_t)	Gas Deposited (g_t)
$t = 1$	?	?
$t = 2$	?	?
$t = 3$	?	?
$\vdots$	$\vdots$	$\vdots$

Now to start, we would like to use our intuition about the problem to come up with some rules which should guide us towards a solution. For instance, its doesn't make any sense that the truck should return home with any fuel left. After all, why not leave it in a cache in the desert where it can do some use?

Problem. Use this idea to come up with a rule relating d_1 and g_1 on the first trip. Are there limits to how large that d_1 and g_1 can be? (What are they?)

Now before we work too hard to get an answer, we might question the second part of the problem, namely "Is there a limit to the width of a desert the truck can cross?" Well, as a first step, lets see if the truck can even cross the desert when it is 800 miles long. A good guess since we are going to work on the problem all week that the answer is "yes." Nonetheless, lets check it.

Problem. Describe a sequence of trips that will let the truck cross a desert which is 800 miles long.

Day 2

We proceed with leading questions.

Problem. Does the nature of the problem change if we scale all distances (namely the length of the desert and the distance a truck can go) by a constant c? Can you think of a convenient scale to choose? Use that scale for the following two questions.

Clearly the maximum distance the truck can go in one trip is 500 miles. But what about the second trip?

Problem. During trip 1, the truck deposits g_1 at distance d_1. Given any g_1 and d_1, how far can the truck go on the second trip as a function of g_1 and d_1?

Now recall the relationship between g_1 and d_1 from Day 1 for the next problem.

Problem. What is the maximum distance (in loads worth of gasoline) that the truck can go in two trips? How much gas is left in the cache from trip one and is there any intuition about a general rule to be had here? Is there such a thing a putting a cache of gas "too close" to where the truck starts to maximize the distance gone in the second trip?

Puzzle (For any day)

Again from Martin Gardner:

> In recent years a number of clever coin-weighing or ball-weighing problems have aroused widespread interest. Here is a new and charmingly simple variation. You have 10 stacks of coins, each consisting of 10 half-dollars. One entire stack is counterfeit, but you do not know which one. You do know the weight of a genuine half-dollar and you are also told that each counterfeit coin weighs one gram more than it should. You may weight the coins on a pointer scale (one which tells you an exact weight). What is the smallest number of weighings necessary to determine which stack is counterfeit?

Hint: The fact that you are allowed to break up the stacks matters, for instance, what is the minimal number of weighings if you are not allowed to break up the stacks?

References

[1] M. Gardner. *My Best Mathematical and Logic Puzzles.* Dover Publications, 1994.