Day 2 (Again, less important questions are marked with ***.)

(1) Rebecca's Problems: 9,10,11, 12 (A only), 13 (A only), 15 (a only) (See answers online).
(2) Day 2 problems on first day sheet. (Ask me in person.)
(3) Find the following quantities for $u = (2, 4, 6)$, $v = (1, 2, 3)$, $w = (0, 1, 0)$
   (a) $\sqrt{v \cdot v} = \sqrt{14}$
   (b) $\sqrt{u \cdot u} = \sqrt{2v \cdot 2v} = 2\sqrt{v \cdot v} = 2\sqrt{14}$
   (c) $u \cdot v = 2v \cdot v = 2 \cdot 14 = 28$
   (d) $w \cdot u = 4$
   (e) $v \cdot u = 2$
(4) Find the maximum of $e^{\alpha x}$ on $[20, 50]$ for $\alpha > 0$. What about for $\alpha < 0$?
The derivative is given by $\alpha e^{\alpha x}$ which is > 0 for $\alpha > 0$ and < 0 for $\alpha < 0$. So for when α is positive, the function is strictly increasing so the maximum occurs at $x = 50$, and when α is negative, the function is strictly decreasing so the maximum occurs at $x = 20$.
(5) Find the maximum of $\alpha x^2 + x$ for $\alpha < -1$ on $[0, 10]$.
For $f(x) = \alpha x^2 + x$ we have

$$\begin{aligned} f'(x) &= 2\alpha x + 1 \\ f''(x) &= 2\alpha < 0 \end{aligned}$$

so f is strictly concave. From the class result, our candidates for a maximum are $0, 10$ and the critical point $-\frac{1}{2\alpha}$ and

$$\begin{aligned} f(0) &= 0 \\ f(10) &= 100\alpha + 10 \\ f(-\frac{1}{2\alpha}) &= \alpha(-\frac{1}{2\alpha})^2 - \frac{1}{2\alpha} \\ &= \frac{1}{4\alpha} - \frac{1}{2\alpha} \\ &= \frac{1}{4\alpha} \end{aligned}$$

Since $\alpha < -1$, $\frac{1}{4\alpha} < 0$ and $100\alpha + 10 < 0$ so the maximum is actually at $x = 0$ which gives $f(0) = 0$.
(6) ***If you're feeling confident (really, it shouldn't be so hard with your notes!). Define $F(x)$ by

$$F(x) = \int_0^{x^2} \frac{1}{2} \ln(y) dy$$

Find the maximum of F on $[1, 100]$ using this definition. Find $F'(x)$ and use it first to see if F is concave, then use it to find a formula for $F(x)$. What is $(xF'(x))'$?
At each $y \in [1, 100]$, the integrand $\frac{1}{2}\ln(y)$ is positive so the larger x is, "the more area" the integral covers to the function is strictly increasing. Thus the maximum occurs at $x = 100$.
Now let $g(x) = x^2$ and

$$h(x) = \int_0^{x} \frac{1}{2} \ln(y) dy$$

so that

$$F(x) = h(g(x))$$

The fundamental theorem of calculus says that

$$h'(x) = \frac{1}{2} \ln(x)$$

and to get $F'(x)$ we can use the chain rule,

$$\begin{aligned} F'(x) &= h'(g(x))g'(x) \\ &= \frac{1}{2}\ln(g(x)) \cdot g'(x) \\ &= \frac{1}{2}\ln(x^2) \cdot 2x \\ &= 2x \cdot \ln x \end{aligned}$$

Once we know this, $(xF'(x))'$ is easy to find, and we also have that

$$F''(x) = 2\ln x + 2$$

and for $x \in [1, 100]$, $F''(x) > 0$ so the function is not concave (but is in fact strictly convex!).

(7) Find the set of all critical points, $\{x : f'(x) = 0\}$, of the following functions (for d-e you should see a trick!):

(a) $f(x) = \alpha$
Critical Points: All of $\mathbb{R}$.

(b) $g(x) = e^{\beta x}$
Critical Points: None! Or rather, the set of critical points is the empty set, $\emptyset$.

(c) $h(x) = ax^3 + bx^2 + cx + d$
$h'(x) = 3ax^2 + 2bx + c$ so to find the critical points, we need to set $h'(x) = 0$, and quadratic (or rather 2nd degree polynomial) so we may use the quadratic formula to get

$$\begin{aligned} x &= \frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a} \\ &= \frac{-b \pm \sqrt{b^2 - 3ac}}{3a} \end{aligned}$$

which has two solutions so long as $b^2 - 3ac > 0$ and one solution if $b^2 - 3ac = 0$, giving two and one critical point, respectively. Finally, when $b^2 - 3ac < 0$ we have no critical points.

(d) ***$f(x) \cdot g(x)$
Here as in (b), we have no criticial points, since the derivative we get is

$$\begin{aligned} (f(x) \cdot g(x))' &= f'(x)g(x) + f(x)g'(x) \\ &= \alpha\beta e^{\beta x} \end{aligned}$$

(e) ***$f(x) \cdot h(x)$
Now we get a new polynomial like in (c), but the coefficients are $\alpha a, \alpha b, \alpha c, \alpha d$ instead of a, b, c, d.