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University of Wisconsin- Madison
Agricultural and Applied Economics
Math Review- August 2005

Prepared by Rebecca Moore

The purpose of the math review is to refresh the math skills should know and will need to know before taking AAE 635. This includes some things taught in Calculus 1 and 2 and Linear Algebra, and a few things not included in those courses. However, in addition to remembering how to take derivatives, find determinants, and optimize functions, it is important to get comfortable reading and writing math. You should have received an email pointing you to the math review notes used in previous years. Many students found these to be either too brief or too advanced. The more math you know, the better off you will be, so if you can work through those notes and problems, do so. However, the practice problems included in the "Long Version" of the notes are similar to homework problems in ECON 703, the mathematical economics course for PhD students. Some questions are basic, but some are very difficult and not necessarily vital topics for success in AAE 635.

I have decided to expand the Brief Version of the notes, to include more explanation and examples, particularly for the material that might be new to you. I also collected a set of practice problems to go with this "Expanded Brief" version. These include some of the practice questions from the long notes, as well as others. The review sessions will focus on these problems.

I. The Basics of Sets and Logic

This stuff may or may not be familiar to you, but it is the basis for how math is written so it's important to be comfortable with it.

Logic

Notation and definitions

- $\equiv$ "is identically equal to" or "is defined as"
- $\exists$ "there exists"
- $\forall$ "for all"
- $\therefore$ "therefore"
- $\Rightarrow$ "implies"
- $\Leftrightarrow$ "if and only if," or "iff" for short
- $\sim$ "not"

The **converse** of $A \Rightarrow B$ is $B \Rightarrow A$. Just because $A \Rightarrow B$ is true, its converse need not be true.
The **contrapositive** of $A \Rightarrow B$ is $\sim A \Rightarrow \sim B$. If $A \Rightarrow B$ is true, its contrapositive must also be true.

Example: A = "x is greater than 5"; B = "x is greater than 3"

In this case, $A \Rightarrow B$, because if $x > 5$ then x must be greater than 3.
The **converse** of this: "$x > 3$" $\Rightarrow$ "$x > 5$" is not true. What if $x = 4$.
The **contrapositive** is: "$x \leq 3$" $\Rightarrow$ "$x \leq 5$" is true.

Necessity and sufficiency

These are two related but very different ideas, and it is important to keep them straight. Consider the example above: $A \Rightarrow B$
B is **necessary** for A; It is necessary that B is true if A is true
A is **sufficient** for B; If A is true, this is sufficient proof that B is also true.

You can have a statement and its converse both be true. Then we can write $A \Leftrightarrow B$, and say A is **necessary and sufficient** for B and B is **necessary and sufficient** for A.

Sets

A **set** is a well-defined collection of elements.

Examples:

$A = \{a, b, c\}$

$J = \{1, 3, 5, 7, \ldots\}$

$K = \{g \mid g \text{ is an odd integer}\}$ (read: "*J* is the set of all *g* such that *g* is an odd integer");

Note that *A* is a set with 3 elements, but *J* and *K* have an infinite number of elements. Also, *J* and *K* are equal to each other ($J = K$)

Notation and terminology:

$a \in A$ "*a* is an element of set *A*"
$4 \notin J$ "4 is not an element of set *J*"
$\varnothing$ or { } the null set or the empty set; the set with no elements
$A \subset B$ or
$B \supset A$ "A is a subset of B"; this means that if $x \in A$, then $x \in B$, for all $x \in A$
$A \cup B$ "A union B"; the set of elements in set A, set B, or both; $A \cup B = \{x: x \in A \text{ or } x \in B\}$.
$A \cap B$ "A intersect B"; the set of elements in both A and B; $A \cap B = \{x: x \in A \text{ and } x \in B\}$.
B\A "the complement of set A in set B"; $B \backslash A = \{x: x \in B \text{ and } x \notin A\}$.
A^c "the complement of set A"; this is in terms of some universal set, like the set of real numbers
A and B are **disjoint sets** if $A \cap B = \varnothing$.

Important sets of numbers

natural or counting numbers: $N = \{1, 2, 3, \ldots\}$
integers: $Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
rational numbers: $Q = \{a/b: a \in Z, b \in Z, b \neq 0\}$
irrational numbers: $2^{1/2}$, $3^{1/2}$, e, π, ...
real numbers: $\mathbf{R} = \{x: x \text{ is rational or irrational}\}$

$\mathbf{R}_+ = \{x: x \in R, x \geq 0\}$

$\mathbf{R}_{++} = \{x: x \in R, x > 0\}$

Elements of sets can be ordered pairs

$S \times T \equiv \{(s,t) \mid s \in S, t \in T\}$ "the set of ordered pairs s, t, such that s is an element of S and t is an element of T"

Examples:

The Cartesian plane, the graphing space we normally use with the x and y axis, can be described as

$$\mathbf{R}^2 \equiv \mathbf{R} \times \mathbf{R} \equiv \{(x_1, x_2) \mid x_1 \in \mathbf{R}, x_2 \in \mathbf{R}\}$$

The northeast quadrant of this space, the "nonnegative quadrant", is then

$$\mathbf{R}^2_+ \equiv \mathbf{R}_+ \times \mathbf{R}_+ \equiv \{(x_1, x_2) \mid x_1 \in \mathbf{R}_+, x_2 \in \mathbf{R}_+\}$$

A three dimensional vector, like (x,y,z), is an element of "3-space"

$$\mathbf{R}^3 \equiv \mathbf{R} \times \mathbf{R} \times \mathbf{R} \equiv \{(x_1, x_2, x_3) \mid x_1 \in \mathbf{R}, x_2 \in \mathbf{R}, x_3 \in \mathbf{R}\}$$

More generally, an n-dimensional vector is an element of "n-space"

$\mathbf{R}^n \equiv \mathbf{R} \times \ldots \times \mathbf{R} \equiv \{(x_1, x_2, \ldots, x_n): x_i \in \mathbf{R}, i = 1, \ldots, n\}$

II. Single variable calculus
This stuff should definitely be familiar. It is generally covered in a 1st semester calculus course.

Functions
A **function** f associates each element of a set X with a unique element in another set Y.
The **domain** of f is all possible elements of set X.
The **range** of f is all possible elements of set Y.

Notation

$f: X \to Y$ (read: "f maps the set X into the set Y.")
f(x) (read: "f of x.")

Often, we assign f(x) a variable name, such as y = 2x. In this case, x is the **independent**, or **exogenous,** variable and y is the **dependent,** or **endogenous,** variable

Examples: $f(x) = x^2$ Domain = $\{x \mid x \in \mathbf{R}\}$, Range = $\{y \mid y \geq 0\}$
or Domain = $(-\infty, +\infty)$, Range = $[0, +\infty)$

The range must be positive because there is no element of the domain that would map into a negative value. In interval notation, the (and) indicate open intervals, mean "not including the endpoints", and the [and] indicate closed intervals, meaning "including the endpoints".

$f(x) = 1/x$ Domain = $\{x \mid x \neq 0\}$, Range = $\{y \mid y \neq 0\}$

Here, the domain is naturally restricted because the function is undefined at x = 0.

$C = 100 + Q$ Domain = $\{Q \mid Q \geq 0\}$, Range = $\{C \mid C \geq 100\}$

If Q is quantity and C is cost, we would want to artificially restrict our domain to only positive values of Q.

Limits
The **limit** of f at b exists and equals A if, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - b| < \delta$, then $|f(x) - A| < \varepsilon$.

Notation

$\lim_{x\to b} f(x) = A$ (read: "the limit of f as x approaches b equals A.")

Notes: f(x) does not have to be defined at b, but must be defined "near" b.
If f(x) is defined at b, it may be the case that $f(x) \neq b$.

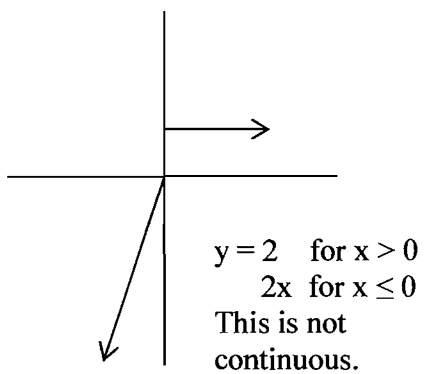
Examples: $f(x) = x^2$
$\lim_{x\to 0} f(x) = 0$ and $f(0) = 0$

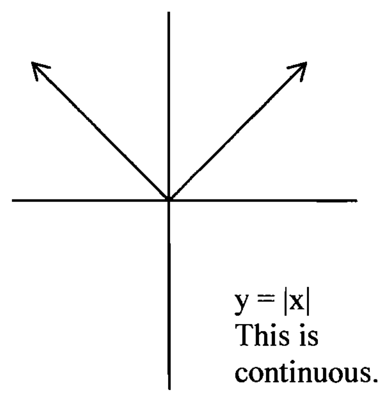
$f(x) = x^2$ for $x \neq 0$
$= -2$, for $x = 0$
Here, $\lim_{x\to 0} f(x) = 0$, but $f(0) = -2$, so the limit as x goes to 0 does not equal f(0)

$f(x) = \ln(x)$
$\lim_{x\to 0} f(x) = -\infty$ but ln(0) is undefined, so the limit exists even though f(0) does not

Continuity
Formal definition: *f* is **continuous** at b if, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - b| < \delta$, then $|f(x) - f(b)| < \varepsilon$. *f* is a continuous function if it is continuous at all points in its domain.

Really: A function is **continuous** if its graph has no breaks





Derivatives
Formal definition: $\partial f(x)/\partial x = \lim_{\Delta x\to 0} [f(x+\Delta x)-f(x)]/\Delta x$
In other words: The **derivative** of a function *f*(x) with respect to its argument x measures the relative change in *f*(x) given a small change in x.

Formal definition: A function is **differentiable** on a set X if and only if the above limit exists for all x in X.
In other words: A differentiable function must be continuous, and its graph must have no kinks. It must be smooth. The graph above on the left is not continuous, so it is not differentiable, either. The absolute value function graphed on the right is continuous, but not differentiable because it has a kink at x = 0.

Notation:
Given y = f(x), the derivative of f with respect to x is written as
$\partial f(x)/\partial x$, $Df(x)$, $f'(x)$, f_x, $\partial y/\partial x$, y', or y_x

The second derivative of f with respect to x, is the derivative of the first derivative, and is written as
$\partial^2 f(x)/\partial x^2$, $D^2 f(x)$, $f''(x)$, f_{xx}, $\partial^2 y/\partial x^2$, y'', y_{xx}

Rules for taking derivatives - You must know these and how to use them
If a and b are constants, and g() and h() are differentiable function

$f(x) = a + bx^c$	$f'(x) = bcx^{c-1}$	
$f(x) = g(x) + h(x)$	$f'(x) = g'(x) + h'(x)$	
$f(x) = g(x) \cdot h(x)$	$f'(x) = g(x) \cdot h'(x) + h(x) \cdot g'(x)$	"product rule"
$f(x) = \dfrac{g(x)}{h(x)}$	$f'(x) = \dfrac{h(x) \cdot g'(x) - g(x) \cdot h'(x)}{[h(x)]^2}$	"quotient rule"
$f(x) = g(h(x))$	$f'(x) = g'(h(x)) \cdot h'(x)$	"chain rule"

Examples:

$$f(x) = x^3 + 2x^2 - 3\sqrt{x}$$

$$f_x = 3x^2 + 4x - \tfrac{3}{2}x^{-(1/2)} = 3x^2 + 4x - \frac{3}{2\sqrt{x}}$$

$$f_{xx} = 6x + 4 + \tfrac{3}{4}x^{-(3/2)}$$

$$f(x) = \frac{x+1}{x-1}$$

$$f_x = \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

$$f_{xx} = \frac{(x-1)^2(0) - (-2)(2)(x-1)(1)}{(x-1)^4} = \frac{4(x-1)}{(x-1)^4} = \frac{4}{(x-1)^3}$$

Properties of single valued functions

Monotonicity: A function f(x) is **monotonic** if f_x has constant sign for all x in X. If f_x is always positive, f is monotonically increasing. If f_x is always negative, f is monotonically decreasing.

(Strict) concavity: A differentiable function f(x) is **concave** on a set X if $f_{xx} \leq 0$ for all x in X
A differentiable function f(x) is **strictly concave** on a set X if $f_{xx} < 0$ for all x in X

(Strict) convexity: A differentiable function f(x) is **convex** on a set X if $f_{xx} \geq 0$ for all x in X
A differentiable function f(x) is **strictly convex** on a set X if $f_{xx} > 0$ for all x in X

Optimization

Standard procedure from calculus class:

To find the maximum or the minimum of the function f(x).

1. Find f'(x)
2. Set f'(x) = 0, and solve for x. This finds all the *critical points,* where the slope of f(x) is zero.
3. For all the critical points found in step 2
   a. If f''(x) > 0 at that point, f(x) has a local minimum at x
   b. If f''(x) < 0 at that point, f(x) has a local maximum at x
   c. If f''(x) = 0 at that point, f(x) has neither a local min nor a local max at x

Now, here's another way to say the same thing, in the language your book will use.

First order necessary conditions (FONC): Suppose f(x) is a continuously differentiable function on a convex set X. Then a **necessary condition** for f to achieve a local extremum (i.e. minimum or maximum) at a point b in X is: f'(b) = 0.

Second order sufficiency conditions (SOSC): Suppose the FONC is satisfied for a point b in X. Then a **sufficient condition** for the point b to be a local maximum (minimum) is: f''(x) <(>) 0 at b.

Second order necessary conditions (SONC): Suppose the FONC is satisfied for a point b in X. Then a **second order necessary condition** for f to achieve a local maximum (minimum) is: f''(x) ≤ (≥) 0 at b.

This says, for f(x) to have a maximum at b, the 1st necessary condition you check is that f'(b) = 0. If this is true, check your second order conditions. If f''(x) < 0 at b, you have sufficient proof that there is a **local maximum** at b.

Example:

Find all extremum of $y = -3x^5 + 5x^3$.

$FONC: y' = 0$

$$\Rightarrow -15x^4 + 15x^2 = 0$$

$$\Rightarrow -15x^2(x^2 - 1) = 0$$

$$\Rightarrow x = -1, 0, 1$$

$SOSC$: check sign of second derivative

$$f''(x) = -60x^3 + 30x$$

$$\Rightarrow f''(-1) = 30 > 0$$

$$\Rightarrow f''(0) = 0$$

$$\Rightarrow f''(1) = -30 < 0$$

The SOSC show there is a local mimimum at x = -1 and a local maximum at x = 1.

Important functions

First, a few rules you might have forgotten but are important:

Manipulating exponents:
$$\begin{cases} x^a x^b = x^{a+b} \quad and \quad \dfrac{x^a}{x^b} = x^{a-b} \\ x^{-a} = \dfrac{1}{x^a} \quad and \quad x^{a/b} = \sqrt[b]{x^a} \\ x^a + x^b \neq x^{a+b}, \text{ this is obvious when you try a simple example.} \\ 2^2 + 2^3 = 4 + 8 = 12 \neq 32 = 2^5 = 2^{2+3} \end{cases}$$

Manipulating logs: $\begin{cases} \ln(ab) = \ln(a) + \ln(b) \\ \ln(a/b) = \ln(a) - \ln(b) \\ \ln(a^b) = b\ln(a) \end{cases}$

Exponential function:

$$f(x) = e^{g(x)}$$

$$f'(x) = g'(x)e^{g(x)}$$

Natural log function:

$$f(x) = \ln(x)$$

$$f'(x) = 1/x$$

Relationship: $x = e^{\ln(x)}$

III. Linear Algebra

This is review material if you've had linear algebra. If you haven't had linear algebra, spend some time to learn at least this stuff, it is very important. The "Long Version" of the notes is much more complete, but this represents the very basics.

Notation:

Linear algebra and matrix notation are just a different way of organizing information.
A matrix is a rectangular set of numbers. For example the matrix A can be written as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

This matrix has m rows and n columns, so the dimensions of this matrix are (m x n).
a_{ij} is the (i, j)-th element of the matrix A, i = 1, 2, ..., m, j = 1, 2, ..., n,

A square matrix has the same number of rows and columns, (*ie.,* m = n),
A symmetric matrix is square and has $a_{ij} = a_{ji}$ for all i, j = 1, ..., n, (*ie.,* it's symmetric about the main diagonal from top-left to bottom-right).
A diagonal matrix is square and has $a_{ij} = 0$ for all $i \neq j$, i, j = 1, 2, ..., n, (*ie.,* only the main diagonal elements can be non-zero)
An identity matrix (denoted by I_n) is a diagonal matrix with $a_{ij} = 1$ for all all i = j (*ie.,* 1's on the diagonal, 0's everywhere else.)
A null matrix (denoted by 0) has if $a_{ij} = 0$ for all i = 1, 2, ..., m, j = 1, 2, ..., n. (*ie.,* 0's everywhere.)

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 7 \\ 3 & 6 & 8 & 9 \\ 4 & 7 & 9 & 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Symmetric Diagonal I_n

A matrix with only one row is a **row vector**; a matrix with only one column is a **column vector**.

Matrix operations

Addition and Subtraction: Individual elements are added or subtracted. Matrices **must** have the same dimensions in order to be added or subtracted, otherwise they are not conformable.

Example:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \quad \begin{bmatrix} 2 \\ 4 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 6 \\ 9 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -6 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 5 & 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} = Can't\ be\ done$$

Matrix multiplication: **Not** done element by element.

Formal definition: Let A be a (m×k) matrix, and B be a (k×n) matrix. Then D = A·B if and only if $d_{ij} = \sum_{s=1}^{k} a_{is}\, b_{sj}$, for all i = 1, 2, ..., m, j = 1, 2, ..., n, where D is a (m×n) matrix.

In symbols:

$$\begin{bmatrix} A & B \\ C & D \\ E & F \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} Aa+Bc & Ab+Bd \\ Ca+Dc & Cb+Dd \\ Ea+Fc & Eb+Fd \end{bmatrix}$$

$$(3x2) \quad (2x2) \qquad (3x2)$$

Helpful hints: Always pay attention to dimensions. The number of columns of the 1st matrix must equal the number of rows of the second.

(k x n) (n x m) = (k x m)

(n x m)(k x n) = not conformable

In general, A · B ≠ B · A (unlike regular multiplication)

A · B = 0 is possible with A ≠ 0 and B ≠ 0. Example:

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}$$

Multilpying A by an identity matrix of the appropriate dimensions gives you A, just like multiplying by 1 in scalar math.

Matrix transpose, A' or A^T

The transpose of the (N×M) matrix A, denoted by A' or A^T, is the (M×N) matrix B satisfying $a_{nm} = b_{mn}$, n = 1, …, N, m = 1, …, M. (*ie.,* flip the matrix along its diagonal).

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, A' = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Determinants, $|A|$

The determinant of a square matrix A, denoted by |A|, is a uniquely defined number associated with that matrix. A matrix is **singular** if |A| = 0. A matrix is **non-singular** if |A| ≠ 0.

Formal Definition: Let $|A_{nm}|$ denote the *(n,m)*th ***minor*** of A, defined as the determinant of the matrix obtained after deleting the n-th row and m-th column from A. The *(n,m)*th ***cofactor*** of A is defined as $C_{nm} = (-1)^{n+m} |A_{nm}|$. The determinant of A is:

$$|A| = \Sigma_{n=1}^{N} a_{1n} \cdot C_{1n}$$

Example:

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 0 & 3 & 2 \\ 3 & 0 & 7 \end{bmatrix}$$

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$
$$= (4)C_{11} + (1)C_{12} + (-1)C_{13}$$

but, $C_{11} = (-1)^{1+1}\begin{vmatrix} 3 & 2 \\ 0 & 7 \end{vmatrix} = 21, \quad C_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 2 \\ 3 & 7 \end{vmatrix} = 6, \quad C_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} = -9$

$so, |A| = (4)(21) + 6 + 9 = 99$

A few shortcuts:

For a (2 x 2) matrix: $If\ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},\ then\ |A| = ad - bc$.

For a (3 x 3) matrix: $If\ A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},\ then\ |A| = aei + bfg + cdh - ceg - fha - ibd$

These just involve multiplying along the diagonals and adding and subtracting these values. For anything larger than a (3 x 3) you need to use the formal definition.

Inverse, A^{-1}

The inverse of A, denoted by A^{-1}, satisfies $A\ A^{-1} = I_N$. **If |A| = 0, the matrix is singular and A^{-1} does not exist.**

Formal definition: $A^{-1} = \frac{1}{|A|}C'$, where C' is the transpose of the cofactor matrix of A defined such that $C_{nm} = (-1)^{n+m}\ |A_{nm}|$, $|A_{nm}|$ denoting the (*n*,m)th minor of A

Example:

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 0 & 3 & 2 \\ 3 & 0 & 7 \end{bmatrix}$$

$$cofactor\ matrix, C = \begin{bmatrix} (-1)^{1+1}\begin{vmatrix} 3 & 2 \\ 0 & 7 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} 0 & 2 \\ 3 & 7 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} 1 & -1 \\ 0 & 7 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 4 & -1 \\ 3 & 7 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 4 & 1 \\ 3 & 0 \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 4 & -1 \\ 0 & 2 \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 4 & 1 \\ 0 & 3 \end{vmatrix} \end{bmatrix}$$

$$\Rightarrow C = \begin{bmatrix} 21 & -(-6) & -9 \\ -7 & 31 & -(-3) \\ 5 & -8 & 12 \end{bmatrix} \Rightarrow C' = \begin{bmatrix} 21 & -7 & 5 \\ 6 & 31 & -8 \\ -9 & 3 & 12 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}C' = \frac{1}{99}\begin{bmatrix} 21 & -7 & 5 \\ 6 & 31 & -8 \\ -9 & 3 & 12 \end{bmatrix}$$

Shortcut for (2 x 2) matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Using matrices to solve systems of equations

Matrix algebra makes it much easier to simultaneously solve systems of linear equations.

For example:

The following system of 2 equations and 2 variables can be written in matrix notation.

$$\left.\begin{matrix} 5x + y = 3 \\ 2x - y = 4 \end{matrix}\right\} \quad \begin{bmatrix} 5 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$(2x2)\ (2x1) = (2x1)$$

Multiplying the matrices to the left of the equal sign will give you the same expressions as in the 2 equations. The matrix equation is in the form **Ax=b**, where **A** is the matrix of coefficients, **x** is the column vector of variables, and **b** is the column vector of constants. The solution to this equation is given by $\mathbf{x}=\mathbf{A}^{-1}\mathbf{b}$, so

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 2 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = -\tfrac{1}{7}\begin{bmatrix} -1 & -1 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = -\tfrac{1}{7}\begin{bmatrix} -7 \\ 14 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

You can verify that when x = 1, and y = -2, both equations above are true.

Generally, we deal with an equal number of linearly independent equations and unknowns (*eg.,* "2 equations, 2 unknowns"), and so A will be a square matrix. As long as $|A| \neq 0$, a *unique* solution will exist. If $|A| = 0$, then there is either no solution, or an infinite number of solutions to the system.

The above process holds generally for systems of *n* equations, although solving a system of more than 3 equations is best left to a machine.

Cramer's Rule- A shortcut for solving systems

If A is non-singular, the determinant of A is non-zero ($|A| \neq 0$), and the solution for the $(N\times1)$ vector $x = (x_1, x_2, \ldots, x_N)'$ can be expressed as follows:

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & \cdots & a_{1M} \\ b_2 & a_{22} & \cdots & a_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ b_N & a_{N2} & \cdots & a_{NM} \end{vmatrix}}{|A|},\ x_2 = \frac{\begin{vmatrix} a_{11} & b_2 & \cdots & a_{1M} \\ a_{12} & b_2 & \cdots & a_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1N} & b_N & \cdots & a_{NM} \end{vmatrix}}{|A|}, \text{ etc}$$

Example (the same problem as earlier, but using Cramer's rule:

$$\begin{bmatrix} 5 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

For the numerator, just take the coefficient matrix,

and replace the appropriate column with the column vector b.

$$\Rightarrow x = \frac{\begin{vmatrix} 3 & 1 \\ 4 & -1 \end{vmatrix}}{\begin{vmatrix} 5 & 1 \\ 2 & -1 \end{vmatrix}} = \frac{(-3)-(4)}{(-5)-2} = \frac{-7}{-7} = 1,\ y = \frac{\begin{vmatrix} 5 & 3 \\ 2 & 4 \end{vmatrix}}{\begin{vmatrix} 5 & 1 \\ 2 & -1 \end{vmatrix}} = \frac{(20)-(6)}{(-5)-2} = \frac{14}{-7} = -2$$

Properties of Matrices

Positive Definite and Negative Definite

These are really important terms in AAE 635 and you will eventually learn more about them. For now, it will be good to practice evaluating the positive and negative definiteness of a matrix. This is kind of complicated, but can be done as follows.

Let $|A_n|$ denote the determinant of the n-th *principal minor* of A, formed by removing the rows (n+1) to N and columns (n+1) to N from A. Note that the *principal minor* is a special type of *minor* like those used to find determinants and inverses. For example, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

there are 3 *principle minors*:

$A_1 = [a_{11}]$, $A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $A_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, etc. Then the criteria for a matrix to be positive definite or negative definite are as follows:

Condition	Criteria	In other words
A is Positive Definite	$\|A_n\| > 0$ for all n = 1, 2, …, N	The determinants of the principal minors are all strictly positive.
A is Positive Semi-Definite	$\|A_n\| \geq 0$ for all n = 1, 2, …, N	The determinants of the principal minors are all non-negative.
A is Negative Definite	$(-1)^n \|A_n\| > 0$ for all n = 1, 2, …, N.	The determinants of the principal minors alternate in sign, starting with $\|A_1\| < 0$
A is Negative Semi-Definite	$(-1)^n \|A_n\| \geq 0$ for all n = 1, 2, …, N.	The determinants of the principal minors alternate in sign, starting with $\|A_1\| \leq 0$

Example:

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 7 \end{bmatrix}, \quad \Rightarrow \quad |A_1| = |2| = 2 > 0, \ |A_2| = \begin{vmatrix} 2 & 3 \\ 3 & 7 \end{vmatrix} = 5 > 0$$

A is positive definite.

$$B = \begin{bmatrix} -3 & 4 \\ 4 & -6 \end{bmatrix}, \Rightarrow \quad |B_1| = |-3| = -3 < 0, \ B = \begin{vmatrix} -3 & 4 \\ 4 & -6 \end{vmatrix} = 2 > 0$$

B is negative definite.

III. Multivariate calculus

This is stuff not usually covered in the first 2 semesters of calculus, but is very important. It is just extending the basic calculus to equations with more than one exogenous variable, which is what we usually have in economics.

Definitions

A multivariate function is just a function with more than one variable, such as:

$y = f(x_1, x_2, x_3)$

$f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$ (the Cobb-Douglas production function)

A **partial derivative** measures the rate of change of a multi-variate function with respect to one of its arguments, *holding the other arguments constant.* For example when taking the partial derivative of $y = f(x_1, x_2, x_3)$ with respect to x_1, treat x_2 and x_3 as constants and proceed as usual. *Notation:* For the function $y = f(x_1, x_2, \ldots, x_m)$, the first partials are written as

$$\partial y/\partial x_i, \ \partial f/\partial x_i, \ f_{xi}, \ f_i.$$

Second partials measure the rate of change of a first partial with respect to one of its arguments, again holding the other arguments constant. Second partials are either "own partials", where the 2^{nd} derivative is taken with respect to the same variable as the 1^{st} derivative, or "cross partials", where the 1^{st} and 2^{nd} derivatives are with respect to different variables. So to find the second derivative f_{12}, first find f_1, then take the derivative of this with respect to the second variable.

The second partials for a function are often arranged in a matrix known as the **Hessian matrix.**

Notation: Own partials: $\partial^2 y / \partial x_i^2$, $\partial^2 f / \partial x_i^2$, f_{x_i,x_i}, f_{ii}

Cross partials: $\partial^2 y / \partial x_i \partial x_j$, $\partial^2 f / \partial x_i \partial x_j$, f_{x_i,x_j}, f_{ij}

Hessian matrix: $\begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{pmatrix}$

Young's Theorem: The main idea of this important theorem is that if the second partials are all continuous, the order of differentiation does not matter. This means that $f_{ij} = f_{ji}$

The **total derivative** of a function $f = f(x_1, x_2, \ldots, x_m)$ measures the actual change in the value of the function given small changes in its arguments:

$$dy = \partial f / \partial x_1 \cdot dx_1 + \partial f / \partial x_2 \cdot dx_2 + \ldots \partial f / \partial x_n \cdot dx_n$$

Some helpful rules for finding partials, these are just extensions of the rules above for single variable functions:

$$y = f(x_1(t), x_2(t), \ldots, x_m(t))$$

$$\partial y / \partial t = f_1\left(\partial x_1 / \partial t\right) + f_2\left(\partial x_2 / \partial t\right) + \ldots + f_m\left(\partial x_m / \partial t\right)$$

$$y = f(x_1, x_2, \ldots, x_m) \cdot g(x_1, x_2, \ldots, x_m)$$

$$y_i = f_i g(\cdot) + g_i f(\cdot)$$

$$y = \frac{f(x_1, x_2, \ldots, x_m)}{g(x_1, x_2, \ldots, x_m)}$$

$$y_i = \frac{g(\cdot) f_i - f(\cdot) g_i}{(g(\cdot))^2}$$

Example:

If $f(x_1, x_2) = x_1 x_2^2 + x_1 x_2$

The first partials are,

$f_1 = x_2^2 + x_2 \;\; and \;\; f_2 = 2x_1x_2 + x_1$

The second partials are,

$$f_{11} = \frac{\partial^2 f}{\partial x_1^2} = 0,\; f_{12} = \frac{\partial^2 f}{\partial x_1 \partial x_2} = 2x_2 + 1,\; f_{21} = \frac{\partial^2 f}{\partial x_2 \partial x_1} = 2x_2 + 1,\; f_{22} = \frac{\partial^2 f}{\partial x_2^2} = 2x_1$$

Note that $f_{12} = f_{21}$ so Young's Theorem holds.

Properties of multivariate functions

Concavity: A differentiable function f(x) is **concave** on a set X iff the Hessian of f(x) is negative semidefinite for all x in X

A differentiable function f(x) is **strictly concave** on a set X iff the Hessian of f(x) is negative definite for all x in X

Convexity: A differentiable function f(x) is **convex** on a set X iff the Hessian of f(x) is positive semidefinite for all x in X.

A differentiable function f(x) is **strictly convex** on a set X iff the Hessian of f(x) is positive definite for all x in X.

Homogeneity: A function is said to be **homogeneous of degree (HOD) r** if and only if $f(t \cdot \mathbf{x}) = t^r \cdot f(\mathbf{x})$.

Example: $f(x_1,x_2) = x_1^2 x_2^2$. Then, $f(t \cdot x) = (t \cdot x_1)^2 (t \cdot x_2)^2 = t^4 \cdot x_1^2 x_2^2 = t^4 \cdot f(\mathbf{x})$. So f is HOD 4.

If a function is **HOD r** and **r > 1**, then the function exhibits **increasing returns to scale**.

If a function is **HOD r** and **r = 1**, then the function exhibits **constant returns to scale**.

If a function is **HOD r** and **r < 1**, then the function exhibits **decreasing returns to scale**.

Euler's Theorem: If f(x) is HOD r, then $rf(x) = xf_x$.

Example: $f(x) = x^2$ is HOD 2. Then, $r \cdot f(x) = 2x^2$ and $x \cdot f_x = x \cdot 2x = 2x^2$.

Optimization

Remember, with y=f(x), we set the 1st derivative equal to 0 and solve for the critical points. Then we evaluate the sign of the second derivative at each critical point. If the second derivative is negative, we have a maximum, if it is positive, we have a minimum.

The process is the same for $y = f(x_1, x_2, \ldots, x_n)$, but each step is more difficult, because there is more than one first derivative, and a whole matrix of second derivatives. Here are the steps for finding maximums and minimums of $y = f(x_1, x_2, \ldots, x_n)$:

1. Find *all* first partials for the function
2. Set all first partials equal to 0, and solve this **system of equations.** Note that linear algebra and Cramer's rule often come in handy here.

Now you have your critical points. These satisfy the **first order necessary conditions (FONC).** You must check the **second order conditions.**

3. Derive the Hessian matrix of 2^{nd} derivatives.
4. The **second order sufficient condition** for a critical point to be a local max is that the Hessian be *negative definite.* The **second order sufficient condition** for a critical point to be a local min is that the Hessian be *positive definite.*

2 examples below.

Example 1:

$f(x,y) = x^3 - y^3 + 9xy$

$FONC: f_x = 0 \Rightarrow 3x^2 + 9y = 0$

$\qquad f_y = 0 \Rightarrow -3y^2 + 9x = 0$

This is a system of 2 non-linear eqns and 2 unknowns, we can solve by substitution.

The first eqn. gives: $y = -\frac{1}{3}x^2$

Substituting into the second eqn gives: $-3\left(-\frac{1}{3}x^2\right)^2 + 9x = 0$

$$\Rightarrow -3\left(\frac{1}{9}x^4\right) + 9x = 0$$

$$\Rightarrow -\frac{1}{3}x^4 + 9x = 0$$

$$\Rightarrow x^4 + 27x = x(x^3 + 27) = 0$$

This has solutions at x = 0 and x = 3. Substituting back into the top equation gives the corresponding y values. So there are critical points at $(x, y) = (0,0)$ and $(x, y) = (3,-3)$.

$SOC: The\ \ Hessian\ \ is:$

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6x & 9 \\ 9 & -6y \end{bmatrix}$$

There are 2 principle minors, and all are positive:

$$A_1 = 6x \qquad A_2 = \begin{vmatrix} 6x & 9 \\ 9 & -6y \end{vmatrix} = -36xy - 81$$

At the point (0,0), $A_1 = 0$ and A_2 = -81 and H is neither positive or negative definite. This is neither a min or a max.

At the point (3,-3), $A_1 = 18 > 0$ and $A_2 = 243 > 0$ and so H is positive definite. There is a local min at (3, -3).

Example 2:

$f(x,y,z) = xz + x^2 - y + yz + y^2 + 3z^2$

$$FONC: f_x = 0 \Rightarrow z + 2x = 0$$
$$f_y = 0 \Rightarrow -1 + z + 2y = 0$$
$$f_z = 0 \Rightarrow x + y + 6z = 0$$

This is a system of 3 linear eqns and 3 unknowns, we can solve by substitution, or with linear algebra.

For practice, I'll use Cramer's rule:

Rearrange the equations:

$2x + z = 0$

$2y + z = 1$

$x + y + 6z = 0$

Write the system in matrix form:

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Using Cramer's rule:

$$x = \frac{\begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 6 \end{vmatrix}}{\begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 6 \end{vmatrix}} = \frac{1}{20} \qquad y = \frac{\begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 6 \end{vmatrix}}{\begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 6 \end{vmatrix}} = \frac{11}{20} \qquad z = \frac{\begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 6 \end{vmatrix}} = \frac{-2}{20}$$

There is a critical point at $(x,y,z) = (1/20, 11/20, -2/20)$

$SOC: The\ Hessian\ is:$

$$H = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 6 \end{bmatrix}$$

There are 3 principle minors, and all are positive:

$$A_1 = 2 \qquad A_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 \qquad A_3 = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 6 \end{vmatrix} = 20$$

There is a local minimum at $(x,y,z) = (1/20, 11/20, -2/20)$

Important note: So far, we have only discussed local min and local max, the optimum in a subset of the domain of the function. These are not necessarily the same as the global min and global max, which is are the optimum for the whole domain. You can worry about the conditions for global extremes when you get to class.

V. What's next

The next level of problems involves optimizing functions subject to a constraint, such as, maximize $(x_1 + x_2)$ such that $1 - x_1x_2 = 0$. This is the type of problem you will see and learn how to solve in AAE 635. The more comfortable you are with the problems in the math review, the less painful that learning process will be.